Order versus Chaos

Mark J.H. van den Bergh
Mathematisch instituut
Leiden University
Leiden, The Netherlands
m.j.h.van.den.bergh@math.leidenuniv.nl

Sipke T. Castelein
Mathematisch instituut
Leiden University
Leiden, The Netherlands
casteleinsipket@gmail.com

Daniël M.H. van Gent
Mathematisch instituut
Leiden University
Leiden, The Netherlands
d.m.h.van.gent@math.leidenuniv.nl

***Abstract*—The positional game of Order versus Chaos can be considered a maker-breaker variant. The players Order and Chaos take turns placing circles or crosses on a board, in which the goal of Order is to create a consecutive line of identical symbols of a certain length, while Chaos aims to prevent this. In this paper, we provide some theoretical results on winning strategies for both players on finite boards of varying sizes, as well as on infinite boards. The composition of these strategies was aided by the use of Monte-Carlo Tree Search (MCTS) players, as well as a SAT solver. In addition to these theoretical results, we provide some more experimental results obtained using MCTS.**

***Index Terms*—Order and Chaos, Positional games, Maker-breaker games, Monte-Carlo Tree Search, Satisfiability**

I. Introduction

The game of Order versus Chaos is a maker-breaker-like positional game [1]. In the original game 'Order and Chaos', as proposed by Stephen Sniderman [2], two players, named Order and Chaos, take turns placing either a circle or a cross on a 6×6 board. Both players are allowed to place either symbol on an empty square. The goal of Order (maker) is to create a horizontal, vertical or diagonal line of (at least) five identical symbols, while the goal of Chaos (breaker) is to prevent this whilst filling the board.

The original version of the game was solved by Benjamin Turner using a brute-force approach [3], showing that Order wins playing first. In this paper, we will discuss a more sophisticated strategy solving the original game. In order to find this strategy, we constructed two artificial players using Monte-Carlo Tree Search (MCTS) simulations, a popular method for solving combinatorial or positional games [4]. Analyzing the moves prescribed for Order by the MCTS algorithm, we distilled an explicit rule-based strategy.

Moreover, we consider larger games, in which the objective for Order is to make a line of more than five in a row. For winning lines of length at least 9, we model our problem as an instance of the Satisfiability (SAT) problem, for which fast solvers are available [5]. We prove constructively that Chaos always wins if Order needs to align at least 10 symbols, and that Chaos wins if Order needs to align 9 symbols and the amount of squares on the board is of suitable parity.

For games in which Order needs a line of length 6, 7 or 8 to win, we prove that Chaos wins if the board is not much larger than the line to be made. Moreover, we use more MCTS simulations to explore these games, conjecturing that these games are winning for Order if and only if the board is large enough.

We start by introducing some notation. Throughout, we denote $[n] = \{1, \ldots, n\}$ for any natural number n. A *board* is a finite set $B \subseteq \mathbb{Z}^2$ and a *game state* of B is a map $B \to S$, where $S = \{\bigcirc, \times, \square\}$ is the set of *symbols* with $\square$ denoting an *empty square*. For $s = \bigcirc$, we define $\overline{s} = \times$, and vice versa. A *line* is a set of the form $\{(x, y) + k \cdot (a, b) \mid 0 \leq k < m\}$ for some $(x, y) \in \mathbb{Z}^2$, $m \in \mathbb{Z}_{>0}$ and non-zero $(a, b) \in \{-1, 0, 1\}^2$, and we call m the length of this line. We call a line L *homogeneous* if either $f[L] = \{\times\}$ or $f[L] = \{\bigcirc\}$, where we write $f[L] = \{f(x) \mid x \in L\}$. The *players* are Order and Chaos.

For a board B, a positive integer m and a player p we define the game $\mathrm{ovc}(B, m, p)$ as follows. The players take turns starting with player p and as initial game state f the empty board, i.e., $f(b) = \square$ for all $b \in B$. If the board is full, i.e., $\square \notin f[B]$, the game ends. Otherwise, a turn consists of choosing some $b \in B$ with $f(b) = \square$ and updating f at b such that $f(b) = \bigcirc$ or $f(b) = \times$. In accordance with the terminology for maker-breaker games, We call a line $L \subseteq B$ of length m a *win line*. We say f is in *order* if there exists a homogeneous win line. If f is in order at the end of the game, then Order wins, and otherwise Chaos wins. The traditional version of Order versus Chaos is thus defined by $\mathrm{ovc}([6]^2, 5, \text{Order})$.

We similarly define the game $\mathrm{ovc}'(B, m, p)$ where the starting player p each turn in addition to his or her usual moves is allowed to pass, i.e., skip their turn. We study this game because it has nice properties with respect to inclusion of boards.

Lemma I.1. *Write $X \preceq Y$ for 'Order wins X implies Order wins Y'. Let $A \subseteq B \subseteq C$ be boards, p a player and $m > 0$. Then*

$$\begin{aligned} \mathrm{ovc}'(A, m, \text{Chaos}) \preceq \mathrm{ovc}'(B, m, \text{Chaos}) \preceq \mathrm{ovc}(B, m, p) \\ \preceq \mathrm{ovc}'(B, m, \text{Order}) \preceq \mathrm{ovc}'(C, m, \text{Order}). \end{aligned}$$

The winning result for Order is summarized as follows.

Theorem I.2. *Let B be a board containing $[n]^2$ for some n. Then Order wins $\mathrm{ovc}'(B, m, \text{Chaos})$ for $(m, n) \in \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 6)\}$.*

978-1-7281-4533-4/20/$31.00 ©2020 IEEE

The result for Chaos for long win lines is as follows.

Definition I.3. For parameters (B, m, p) we say the game $\mathrm{ovc}(B, m, p)$ has *good parity* if $|B|$ is even when p is Order and $|B|$ is odd when p is Chaos.

Theorem I.4. *Let B be a board and let p be a starting player. Then*

(i) *Chaos wins $\mathrm{ovc}(B, m, p)$ for all $m \geq 9$ when the game has good parity.*

(ii) *Chaos wins $\mathrm{ovc}'(B, m, \text{Order})$ for all $m \geq 10$.*

Clearly, good parity can be obtained by passing the first turn, so assuming Theorem I.4 we have that Chaos wins $\mathrm{ovc}'(B, m, \text{Chaos})$ for all boards B and $m \geq 9$.

For small boards, we have the following result for Chaos.

Proposition I.5. *Chaos wins $\mathrm{ovc}'([5 + 2m]^2, 5 + m, \text{Order})$ for all $m \geq 0$.*

We prove Lemma I.1 and discuss some more subtleties considering passing in Section II. In Section IV, we constructively prove Theorem I.2, using a strategy inspired by MCTS play. This explicit rule-based strategy leads to a win for Order starting from the empty board, therewith weakly solving the game. Note that this contrasts the solution in [3], where a winning move is listed for every winning position of Order, strongly solving the game.

The proof of Theorem I.4 can be found in Section III, resting on a SAT formulation of the game. In this section, we also prove Proposition I.5. For some of the games not covered by the above theorems, we provide MCTS simulations in Section V. Finally, we discuss some generalizations of these results to infinite boards in Section VI.

Throughout, in all strategies discussed, the listed steps are executed in order. When a move is made, after the other player's turn, we start again with the execution of the first step.

II. Maker-breaker games and passing

A classic maker-breaker game is defined by a set U and a family of winning sets $\mathcal{F} \subseteq 2^U$. Maker and breaker alternate turns, each turn claiming an unclaimed element from U. Maker wins by claiming all elements in some $F \in \mathcal{F}$, while breaker wins by claiming at least one element from each $F \in \mathcal{F}$.

The Order versus Chaos game $\mathrm{ovc}(B, m, p)$ differs from classic maker-breaker games in that both players do not claim an element from B as their own on their turn, but instead assign an as of yet unassigned element to either $X = \{b \in B \mid f(b) = \times\}$ or $O = \{b \in B \mid f(b) = \bigcirc\}$. The winning sets are the win lines, and Order (maker) wins if either $L \cap X = L$ or $L \cap O = L$ for some win line L, while Chaos (breaker) wins if $L \cap X \neq \emptyset$ and $L \cap O \neq \emptyset$ for all win lines L.

While it is clear that in a classic maker-breaker game, it is disadvantageous to pass for either player, this is not always true in Order versus Chaos.

Example II.1. Consider $\mathrm{ovc}([2] \times [1], 2, \text{Chaos})$. It is clear that Order wins this game. However, the variant $\mathrm{ovc}'([2] \times [1], 2, \text{Chaos})$ is won by Chaos by passing on the first turn. Similarly, the game $\mathrm{ovc}([2] \times [1], 2, \text{Order})$ is won by Chaos, while $\mathrm{ovc}'([2] \times [1], 2, \text{Order})$ is won by Order.

Some relations between the passing variant and the regular game are summarized in the following lemma.

Lemma I.1. *Write $X \preceq Y$ for 'Order wins X implies Order wins Y'. Let $A \subseteq B \subseteq C$ be boards, p a player and $m > 0$. Then*

$$\begin{aligned}\mathrm{ovc}'(A, m, \text{Chaos}) \preceq \mathrm{ovc}'(B, m, \text{Chaos}) \preceq \mathrm{ovc}(B, m, p) \\ \preceq \mathrm{ovc}'(B, m, \text{Order}) \preceq \mathrm{ovc}'(C, m, \text{Order}).\end{aligned}$$

Proof. To prove the third inequality, if Order wins $\mathrm{ovc}(B, m, \text{Order})$, it is clear that Order also wins $\mathrm{ovc}'(B, m, \text{Order})$. If Order wins $\mathrm{ovc}(B, m, \text{Chaos})$, then Order wins $\mathrm{ovc}'(B, m, \text{Order})$ by using the same strategy, but passing on the first turn.

For the second inequality, we use a similar argument, but by contrapositive. Hence, suppose that Chaos wins $\mathrm{ovc}(B, m, p)$ for some p. Then Chaos must also win $\mathrm{ovc}'(B, m, \text{Chaos})$ using the same strategy if $p = $ Chaos, and passing on the first turn if $p = $ Order.

For the first inequality, Order can apply her winning strategy for A to B by treating every move by Chaos outside of A as a pass from Chaos on A.

For the last inequality, suppose Order has a winning strategy for $\mathrm{ovc}'(B, m, \text{Order})$. Then she can win $\mathrm{ovc}'(C, m, \text{Order})$ by applying this strategy when Chaos moves in B while passing when Chaos moves in $C \setminus B$. $\square$

Example II.1 shows that the implications defined by the second and third inequalities are not equivalences. The next example shows that holds for both other implications, as well.

Example II.2. Consider $\mathrm{ovc}'([1]^2, 2, \text{Chaos})$ and $\mathrm{ovc}'([2]^2, 2, \text{Chaos})$. It is clear that Order cannot win $\mathrm{ovc}'([1]^2, 2, \text{Chaos})$, as there is not enough space for a line of length 2, while it is straightforward to check that Order wins $\mathrm{ovc}'([2]^2, 2, \text{Chaos})$. Similarly, Order wins $\mathrm{ovc}'([2]^2, 2, \text{Order})$, but not $\mathrm{ovc}'([1]^2, 2, \text{Order})$.

Remark II.3. In ovc', a player never needs to pass on consecutive turns. Suppose a player p passes at turn n, the opposing player makes a move s at b, and then passing is a winning move for p. Instead player p can play s at b turn n, resulting in the same winning game state. Hence the pass was unnecessary.

III. Winning strategies for chaos

We prove Theorem I.4 by explicitly constructing a strategy for Chaos. In the construction, we use a so-called "pairing strategy", the likes of which can be used to solve, e.g., variants of tic-tac-toe, as well as a variant of the original version of Order versus Chaos [1], [6].

Throughout, $B \subseteq \mathbb{Z}^2$ will be a board. For a line $L \subseteq \mathbb{Z}^2$ of length $m \geq 2$ there exist two lines L_+ of length $m + 1$ such that $L \subseteq L_+$. We choose $L_+ = L \cup \{(x, y)\}$ such that $2y - x$ is maximal among the two possibilities, or equivalently such that (x, y) occurs before the points of L in 'reading order' (left-to-right, top-to-bottom).

Proposition III.1. *There exists a partitioning $\mathcal{P}$ of $\mathbb{Z}^2$ into lines of length 2 such that the following holds:*

(i) *For every line L of length 9 there exists a $P \in \mathcal{P}$ such that $P \subseteq L$.*

(ii) *For every line L of length 10 there exists a $P \in \mathcal{P}$ such that $P_+ \subseteq L$.*

We call $\mathcal{P}$ a 'pairing' and its elements 'pairs'.

Proof. We give a constructive proof.

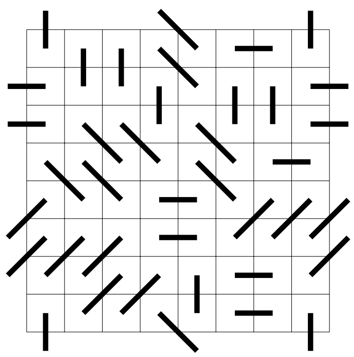

Figure 1: Pairing for good parity

Consider Figure 1, where each square in the grid represents an element of the flat torus $(\mathbb{Z}/8\mathbb{Z})^2$, and each thick line denotes a pair of two adjacent squares. Observe that every row, column and diagonal of $(\mathbb{Z}/8\mathbb{Z})^2$ contains a pair $\overline{P}$.

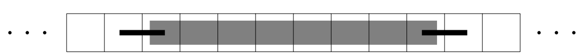

Figure 2: From $(\mathbb{Z}/8\mathbb{Z})^2$ to $\mathbb{Z}^2$

Equivalently, Figure 1 gives a partitioning $\mathcal{P}$ of $\mathbb{Z}^2$ into lines of length 2 by daisy chaining the pattern. For any line $L \subseteq \mathbb{Z}^2$ of length at least 8 we consider its image $\overline{L}$ in $(\mathbb{Z}/8\mathbb{Z})^2$ and note that this image must contain a pair $\overline{P}$. If L has length exactly 8, it is possible that L intersects two pairs with image $\overline{P}$ non-trivially without containing any, as in Figure 2 where L is drawn grey. However, in this case, extending L by one in either direction solves this problem, which proves (i). Proving (ii) goes similarly. $\square$

The pairing given by Figure 1 is rather irregular and hard to find. One can easily find pairings for $(\mathbb{Z}/n\mathbb{Z})^2$ for $n > 8$. The reason for this is straightforward combinatorics: there are $4n$ lines that have to contain a pair, which requires $8n$ points, while we have n^2 points available. This also suggests that a pairing for $n = 8$ could just be possible.

To find the pairing in Figure 1, we formulated the problem as an instance of the Satisfiability (SAT) problem. An instance of SAT consists of a Boolean expression in conjunctive normal form, and a solution is a true/false assignment of the variables that makes the expression true, or a proof that such an assignment does not exist. While the SAT problem has long been known to be NP-complete, modern-day solvers can still efficiently solve sizeable instances with ease. To find the required pairing, we have used the PicoSAT solver [7].

We introduce a variable $x_{\{p,q\}}$ for each pair of adjacent points $p, q \in (\mathbb{Z}/8\mathbb{Z})^2$. Setting $x_{\{p,q\}}$ to true corresponds to pairing the squares p and q. For every two intersecting pairs $A \neq B$, we add a clause $(\neg x_A) \vee (\neg x_B)$ to guarantee that a square is paired to at most one other square. Now, any line $L \subseteq (\mathbb{Z}/8\mathbb{Z})^2$ of length 8 contains 8 pairs of adjacent points $A_1, \ldots, A_8$, of which at least one pair must be coupled. To do so, we add a clause $x_{A_1} \vee \ldots \vee x_{A_8}$. Any solution to the conjunction of the aforementioned clauses thus corresponds to a pairing as desired, and the pairing in Figure 1 is such a solution.

Using this pairing, we now describe a strategy for Chaos.

Strategy III.2. Let $\mathcal{P}$ be a pairing of $\mathbb{Z}^2$ given by Proposition III.1. For $b \in \mathbb{Z}^2$, write $\overline{b}$ for the unique element such that $\{b, \overline{b}\} \in \mathcal{P}$. Let f be the current state, $E = f^{-1}[\{\square\}]$ the set of empty squares and $U = \{b \in B \mid \overline{b} \notin B\}$ the set of unmatched squares.

(i) If, in the previous turn, Order played $s \in S$ at $b \in B$ such that $\overline{b} \in E$, then play $\overline{s}$ at $\overline{b}$.

(ii) If there exists some $b \in E \cap U$, then play anything at b.

(iii) Choose any $b = (x, y) \in E$ such that $2y - x$ is maximal and let $c \in \mathbb{Z}^2$ be such that $\{b, \overline{b}\}_+ = \{b, \overline{b}, c\}$. If $c \in B$, play $\overline{f(c)}$ at b. Otherwise, play anything at b.

Theorem I.4. *Let B be a board and let p be a starting player. Then*

(i) *Chaos wins* $\mathrm{ovc}(B, m, p)$ *for all $m \geq 9$ when the game has good parity.*

(ii) *Chaos wins* $\mathrm{ovc}'(B, m, \mathrm{Order})$ *for all $m \geq 10$.*

Proof. We show that Strategy III.2 is well-defined and winning for Chaos. Note that in Strategy III.2, a move at b is only made when $f(b) = \square$, i.e., all moves are legal. In step (iii), note that $f(c) \neq \square$ when $c \in B$, otherwise we would have chosen c instead of b as the square to make our move in. Hence Strategy III.2 is well-defined.

(i) First, we consider $\mathrm{ovc}(B, 9, p)$ with good parity. Note that in this case Chaos will always be last to play in U: if Chaos is the starting player, then $|U|$ is odd and he plays in U his first turn; if Order is the starting player, then $|U|$ is even. Chaos plays in U when Order did, so the last turn $|U \cap E|$ will always be even at the start of Order's turn. Consequently, we never enter step (iii) of Strategy III.2 when the game has good parity. In this case, when the game ends, we have for each $b \in B$ that either $b \in U$ or $\overline{f(b)} = f(\overline{b})$ by step (i) and (ii) of the strategy. Then, by Proposition III.1, every win line contains a pair $\{b, \overline{b}\}$, which we just noted is not homogeneous. Hence the board is not in order and Chaos wins, proving (i).

(ii) Now consider the game $\mathrm{ovc}'(B, 10, \mathrm{Order})$. For $P \in \mathcal{P}$ such that $P_+ \subseteq B$, we consider the first time a player plays at P. If Order is first to play in P, then Chaos follows in step (i), after which P and in particular P_+ becomes non-homogeneous. When Chaos is first to play in P, then this must happen in step (iii), after which P_+ becomes non-homogeneous. Then, at the end of the game, by Proposition III.1, every win line contains a P_+ for some $P \in \mathcal{P}$, none of which are homogeneous. Hence again Chaos wins.

The rest of the statement now follows from Lemma I.1. $\square$

Strategy III.2 shows that the game is winning for Chaos if Order needs to make a long homogeneous line to win. For shorter win line length, Proposition I.5, which we prove next, gives some specific results.

Proposition I.5. *Chaos wins* $\mathrm{ovc}'([5+2m]^2, 5+m, \mathrm{Order})$ *for all* $m \geq 0$.

Proof. We begin by showing that Chaos wins $\mathrm{ovc}([5]^2, 5, \mathrm{Order})$ similarly to Theorem I.4.ii, namely by partitioning the board so that some of the squares are matched. Consider the (partial) pairing $\mathcal{P}$ as displayed in the center 5×5 subboard B of Figure 3, where squares with the same number are paired and non-numbered squares remain unpaired. Again, for $b \in \mathbb{Z}^2$, we write $\overline{b}$ for the unique element such that $\{b, \overline{b}\} \in \mathcal{P}$. Note that every line $L \subseteq B$ of length 5 contains a pair. We now give a slight modification of Strategy III.2 for Chaos.

			40	41		41	42			
			30	31		31	32			
			20	21		21	22			
45	35	25	3	1	6	1	0	23	33	43
47	37	27	10	2	5	5	4	24	34	44
			9	2		8	9			
47	37	27	10	11	11	8	4	24	34	44
42	32	22	0	7	6	7	3	20	30	40
			23	26		26	25			
			33	36		36	35			
			43	46		46	45			

Figure 3: (Partial) pairing for $[5]^2$ and $[11]^2$

(i) If in the previous turn Order played s at b such that $\{b, \overline{b}\} \in \mathcal{P}$ and $\overline{b}$ is empty, play $\overline{s}$ at $\overline{b}$.
(ii) If there is an empty $b \in B$ such that there is no empty $\overline{b} \in B$, play anything at b.
(iii) If there is a pair $\{b, \overline{b}\} \in \mathcal{P}$ with b empty such that the corresponding win line contains $s \in \{\times, \bigcirc\}$, play $\overline{s}$ at b.

Analogous to the proof of Theorem I.4, this is a winning strategy for Chaos if step (iii) is well-defined, i.e., if we can always find such a pair. We enter step (iii) only when the center is filled, in which case we can play in the pair marked 0, assuming it is not already filled. Note that every pair numbered $n \geq 1$ has in its associated win line a square of the pair numbered $n-1$. Therefore, inductively, we can always play in the pair marked n with n minimal among the empty pairs. Hence Chaos wins $\mathrm{ovc}'([5]^2, 5, \mathrm{Order})$.

Note that only $m = 1, 2, 3, 4, 5$ remain, as the rest follows from Theorem I.4. We give a proof for $m = 3$ as the rest goes analogously and is left as an exercise for the reader. For the board $[11]^2$ we apply the previous strategy to the center 5×5 squares. Then note that almost all lines of length 8 have 5 squares in the center subboard and are already taken care of. The only extra lines are a few off-diagonals, and the orthogonal lines contained completely in the border, and, if we label the border as in Figure 3, all of them contain pairs. When the unmatched squares and the center subboard are completely filled, each of the remaining pairs on the border has a filled square in between them, so we may block the corresponding lines as in step (iii) of the strategy for $[5]^2$. Hence Chaos wins $\mathrm{ovc}'([11]^2, 6, \mathrm{Order})$.

To generalize this to other m one needs to extend or restrict the given border for the 11×11 board in the obvious way. $\square$

A different approach to designing strategies for maker-breaker games is using a potential function, which assigns a value in $[0, 1]$ to every win line [8]. A line containing both symbols is assigned the value 0, a homogeneous line is mapped to 1 and any other win line is assigned a value non-decreasing in the amount of empty squares. If the total potential of all win lines is strictly less than 1 at the start of the game, to show that Chaos wins playing first, it suffices to show that after every pair of moves of Chaos and Order, the total potential has not increased. While this is a straightforward argument for true maker-breaker games, it is hard for Order versus Chaos, as a move by Chaos can increase the potential gained by a subsequent move of Order.

IV. Winning strategies for Order

We continue by proving Theorem I.2, first taking care of the small cases.

Lemma IV.1. *Order wins* $\mathrm{ovc}'([n]^2, n, \mathrm{Chaos})$ *for* $n = 1, 2, 3$.

Proof. For $n = 1, 2$ this is trivial, so consider $n = 3$. If Chaos chooses to move we may assume by rotating the board that he moves anywhere in the lower triangular subboard B colored white in Figure 4. If Chaos passes we may move at the square marked with a dot. Then, regardless of whether Chaos passes the next turn we may apply rotations to the board such that precisely one square of B is filled and it is Order's turn.

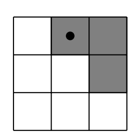

Figure 4: 3×3 board

Now Order can force Chaos to keep playing in B by repeatedly forming a line of length 2. It can easily be verified that this always results in a win for Order. $\square$

We continue by assessing $\mathrm{ovc}'([4]^2, 4, \mathrm{Chaos})$. For this game, we constructed a player for both Order and Chaos using Monte-Carlo Tree Search (MCTS) with Upper Confidence Bounds applied to Trees (UCT) as selection method [9], [10], further explained in Section V. Pitting these MCTS players against each other, we analyzed the strategy employed by Order in a myriad of games. From this analysis, an explicit rule-based strategy for Order was distilled [11].

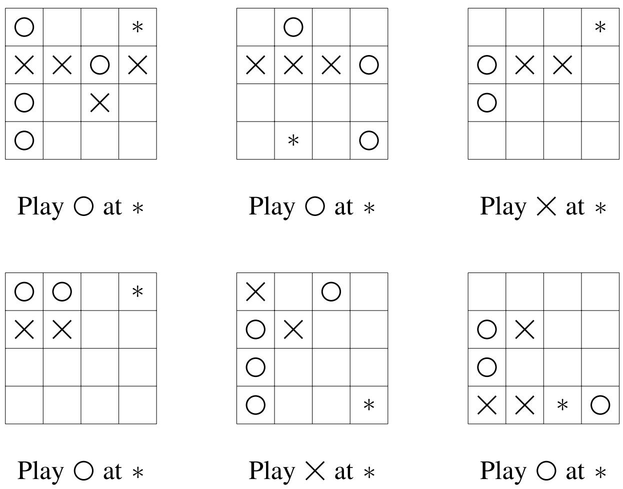

Figure 5: Exceptions

Strategy IV.2. Let $B = [4]^2$, f the current state and $E = f^{-1}[\{\square\}]$ the set of empty squares. Let $\mathcal{L}$ be the set of win lines, and for $b \in B$, let $\mathcal{L}_b = \{L \mid b \in L \in \mathcal{L}\}$. We call $L \in \mathcal{L}$ *broken* when $\{\bigcirc, \times\} \subseteq f[L]$ and *unbroken* otherwise. For an unbroken line L we define its *weight* $w(L) = |\{b \in L \mid f(b) \neq \square\}|$.

Whenever there is a choice between playing in different $b \in B$, we pick the lexicographically minimal, where the northwest square is numbered $(1, 1)$. In addition to this, there are six exceptional boards. These boards and the corresponding winning moves for Order are in Figure 5.

(i) If the board is in Figure 5, play the defined move.
(ii) If $f[B] = \{\square\}$, play $\times$ at $(2, 2)$.
(iii) If there exists an unbroken $L \in \mathcal{L}$ with $w(L) = 3$, we win by playing in L.
(iv) If there exist a $b \in E$ and distinct unbroken $L_1, L_2 \in \mathcal{L}_b$ of weight 2 such that $f[L_1] = f[L_2] = \{\square, s\}$ for some $s \in \{\bigcirc, \times\}$, play s at b.
(v) Let $\mathcal{E}_s = \{b \in E \mid (\forall L \in \mathcal{L}_b)\ \overline{s} \in f[L] \Rightarrow s \in f[L]\}$ be the set of squares in which playing s does not break a line. If $\mathcal{E} = \mathcal{E}_\times \cup \mathcal{E}_\bigcirc$ is non-empty, let L be a win line intersecting $\mathcal{E}$ of maximal weight. For any $b \in L \cap \mathcal{E}_s$ for some $s \in \{\bigcirc, \times\}$, play s at b.
(vi) Let $\mathcal{L}_s$ be the set of win lines containing $s \in \{\bigcirc, \times\}$ and let $\mathcal{L}_s^*$ be the set of lines in $\mathcal{L}_s$ of maximal weight w^* among all unbroken lines. For any square b, let $w_s(b) = \max\{w(L) \mid b \in L \in \mathcal{L}_s\}$ be the weight of the longest unbroken line through b. We play s in b as to maximize $w_s(b)$ under the constraint that there is no $L \in \mathcal{L}_{\overline{s}}^*$ for which $b \in L$, so that we do not break any line of weight w^*.

For illustration, we draw what will happen when we encounter exception 4. The number above the symbol is the turn in which the symbol was played, and the letter below identifies the player that makes this move. Note that, to verify that the strategy is winning for Order, we need to check all possible moves of Chaos. Here, we show only two, as means of example.

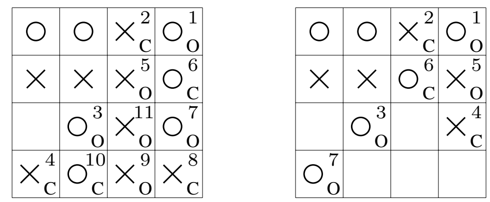

Figure 6: Working out two possible outcomes of Exception 4

Lemma IV.3. *Order wins* $\mathrm{ovc}'([4]^2, 4, \mathrm{Chaos})$.

Proof. We verify that Strategy IV.2 is winning for Order by straightforward computer proof: we check that the strategy is weakly winning against a brute-force player for Chaos. See the Appendix for a reference to the source code. $\square$

To complete the proof of Theorem I.2, we provide the following lemma, based on the result in [3], solving the final case $\mathrm{ovc}'([6]^2, 5, \mathrm{Chaos})$.

Lemma IV.4. *Let* $n \geq 1$. *If Order wins* $\mathrm{ovc}'([n]^2, n, \mathrm{Chaos})$, *then Order also wins* $\mathrm{ovc}'([n+2]^2, n+1, \mathrm{Chaos})$.

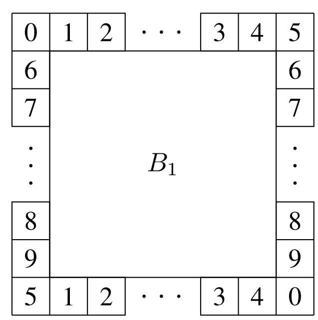

Figure 7: Mirroring strategy on B

Proof. We partition $B = [n+2]^2$ into its $n \times n$ center B_1 and its border $B_2 = B \setminus B_1$. As in Figure 7 we consider the pairing $\mathcal{P}$ of B_2 that pairs opposing squares, i.e., $\{u, v\} \in \mathcal{P}$ if and only if $u, v \in B_2$ are distinct and $\{u, v\} \subseteq L$ for some line $L \subseteq B$ of length $n + 2$ intersecting B_1. We consider the following strategy for Order:

(i) If we can win by completing a win line, do so.
(ii) If Chaos plays in B_2, play the opposing symbol in the paired square.

(iii) Apply the winning strategy for $\mathrm{ovc}'(B_1, n, \text{Chaos})$ to B_1.

We show that the strategy is well-defined and winning. Note that we only play in B_2 in response to Chaos in step (ii) or when we win in step (i). Hence if Chaos plays in B_2, then B_2 will always contain an odd number of filled squares and since $|B_2| = 4(n+1)$ is even there must be an empty square left. Thus step (ii) is well-defined. If B_1 is filled at the start of our turn, then due to the strategy applied it contains a homogeneous line L of length n. Then there exists a pair $P \in \mathcal{P}$ such that $L \cup P$ is a line of length $n+2$. We either have that P is empty, in which case we can in fact win in step (i), or both squares are filled with opposing symbols due to step (ii), in which case a homogeneous line of length $n+1$ already exists. Hence step (iii) is well-defined, and since B_1 will be filled at some point in the game, the strategy is also winning. □

Theorem I.2. *Let B be a board containing $[n]^2$ for some n. Then Order wins $\mathrm{ovc}'(B, m, \text{Chaos})$ for $(m, n) \in \{(1,1), (2,2), (3,3), (4,4), (5,6)\}$.*

Proof. Combine Lemma IV.1, Lemma IV.4 applied to Lemma IV.3, and Lemma I.1. □

V. Monte Carlo results

Monte-Carlo Tree Search (MCTS) is an algorithm that, for every move, iteratively builds a game tree in search of good states, starting with the current state as root [10]. The search is done in four steps per iteration.

(i) *Selection.* In the current state s, we select a state t reachable by one move which has not been visited yet. If such a state does not exist, we pick a reachable state t for which
$$v_t + C\sqrt{\frac{\ln n_s}{n_t}}$$
is maximized, where n_s is the amount of times state s has been visited (analogous for n_t), v_t is the percentage of visits that eventually led to a win for the current player, and C is a chosen constant. We continue selecting until we reach an as-of-yet unvisited state or a terminal state in which either player has won.

(ii) *Play-out.* The game is finished by making random moves until a player has won.

(iii) *Expansion.* The new state which was encountered for the first time is added to the tree.

(iv) *Backpropagation.* All the states that have been visited in the current iteration are updated to incorporate the results of the played-out game.

After a set amount of iterations, cq. play-outs, the current player performs the move leading to the state t with the highest v_t, and the search is continued.

To find Strategy IV.2, the algorithm was run with $C = \sqrt{2}$ and 5,000 play-outs per move. After running the algorithm numerous times, we discovered a pattern in Order's moves, which was used to synthesize the steps of the strategy. While the first two rules of Strategy IV.2 are straightforward, the latter ones are more involved. By testing the algorithm without the exceptional step (i), the game was weakly solved except for in the six cases shown in Figure 5. Adding these boards as exceptions, the strategy weakly wins the game for Order.

# itr.	100,000									60,000	
$m \diagdown n$	3	4	5	6	7	8	9	10	11	12	13
3	100	100	100	-	-	-	-	-	-	-	-
4	-	100	100	100	-	-	-	-	-	-	-
5	-	-	0	91	99	100	100	-	-	-	-
6	-	-	-	0	0	22	50	91	99	100	-
7	-	-	-	-	0	0	0	0	1	23	64
8	-	-	-	-	-	-	-	-	0	0	1

Table I: Win percentage of Order in MCTS simulation of one hundred $\mathrm{ovc}([n]^2, m, \text{Order})$ games.

# itr.	100,000									60,000	
$m \diagdown n$	3	4	5	6	7	8	9	10	11	12	13
3	100	100	100	-	-	-	-	-	-	-	-
4	-	100	100	100	-	-	-	-	-	-	-
5	-	-	0	95	100	100	100	-	-	-	-
6	-	-	-	0	0	18	55	90	100	100	-
7	-	-	-	-	0	0	0	0	2	27	61
8	-	-	-	-	-	-	-	-	0	0	0

Table II: Win percentage of Order in MCTS simulation of one hundred $\mathrm{ovc}([n]^2, m, \text{Chaos})$ games.

# itr.	100,000									60,000	
$m \diagdown n$	3	4	5	6	7	8	9	10	11	12	13
3	100	100	100	-	-	-	-	-	-	-	-
4	-	100	100	100	-	-	-	-	-	-	-
5	-	-	0	86	100	100	100	-	-	-	-
6	-	-	-	0	0	15	61	92	99	100	-
7	-	-	-	-	0	0	0	0	1	14	50
8	-	-	-	-	-	-	-	-	0	0	7

Table III: Win percentage of Order in MCTS simulation of one hundred $\mathrm{ovc}'([n]^2, m, \text{Order})$ games.

# itr.	100,000									60,000	
$m \diagdown n$	3	4	5	6	7	8	9	10	11	12	13
3	100	100	100	-	-	-	-	-	-	-	-
4	-	100	100	100	-	-	-	-	-	-	-
5	-	-	0	95	100	100	100	-	-	-	-
6	-	-	-	0	0	22	52	92	98	100	-
7	-	-	-	-	0	0	0	0	1	33	67
8	-	-	-	-	-	-	-	-	0	0	0

Table IV: Win percentage of Order in MCTS simulation of one hundred $\mathrm{ovc}'([n]^2, m, \text{Chaos})$ games.

For the game of $\mathrm{ovc}([n]^2, m, p)$, we have exhaustive theoretical results for $m \leq 5$ in Theorem I.2 and for $m \geq 10$ in Theorem I.4. However, for $m \in \{6, 7, 8, 9\}$, our only provable result is for small boards in Proposition I.5, and for boards of good parity in Theorem I.4. For these values of m, we conducted additional MCTS experiments.

The results of the experiments can be found in Table I and Table II. The algorithm was run with $C = \sqrt{2}$ for all values of m and n. The amount of play-outs allowed per move is dependent on n and shown in the first row of the tables. Each game was simulated 100 times; the tables show the amount of games won by Order.

Although we must note that it is unknown whether MCTS inherently favors one of the asymmetric players, we formulate some conjectures looking at the results. For $m = 6$, we know that the game on the board $[n]^2$ is winning for Chaos for $n \leq 7$, which is also the result found by MCTS. For $n = 8$, the game also seems to be winning for Chaos. For $n = 9$, the simulation results are unclear. For $n \geq 10$, the game appears to be winning for Order.

For $m = 7, 8$, MCTS again shows an increase in win rate for Order when the board becomes larger, leading to the conjecture that $\operatorname{ovc}([n]^2, m, p)$ is winning for Order if and only if $n \geq N_m$ for some fixed N_m. However, the results do not show a sharp threshold with the current amount of iterations, and thus do not give an indication for the values of N_m.

In Section II, we remarked that, theoretically, passing may be a beneficial move for either player. To investigate the impact of allowing a player to pass, we ran MCTS simulations for the game ovc', the results of which can be found in Table III and Table IV.

Note that the results are very similar to those in Table I. Hence, it appears that allowing the starting player to pass does not have a large impact; investigating the moves made by the MCTS players, we find that passing is rarely done. It thus seems that, in the games we consider here, passing is not advantageous to either player. Finally, note that for these games, also the starting player does not seem to have a noticeable effect.

VI. Infinite boards

We can generalize $\operatorname{ovc}(B, m, p)$ and $\operatorname{ovc}'(B, m, p)$ to non-finite boards B. Here, Order wins as usual when there is a homogeneous line of length m. As there is no way to fill the board, Chaos cannot win in the traditional sense. Hence, we define $\operatorname{ovc}(B, m, p)$ for infinite B to be won by Chaos if no winning strategy for Order exists.

First note that Lemma I.1 generalizes to infinite boards almost perfectly. The only exception is $\operatorname{ovc}'(B, m, \text{Order}) \preceq \operatorname{ovc}'(C, m, \text{Order})$, which only holds when $C \setminus B$ is finite, as Order wants to ensure they do not keep playing in $C \setminus B$ forever when Order plans to win on B. Now Theorem I.2 is applicable and Order wins $\operatorname{ovc}'(B, m, \text{Chaos})$ for small m when B contains a sufficiently large square subboard. For $m = 2$ we can do slightly better.

We say $a, b \in B$ are *neighbours* if $\{a, b\}$ is a line of length 2. The *connected* relation is then the transitive closure of the neighbour relation.

Lemma VI.1. *Let B be a possibly infinite board. Then Order wins $\operatorname{ovc}'(B, 2, \text{Chaos})$ if and only if B contains a connected component of size at least 3.*

Proof. Note that a connected component of size at least 3 contains a pair of distinct intersecting win lines L_1 and L_2. Since Chaos can never play in an empty win line, Order can play in $L_1 \setminus L_2$ and win in her next turn. If all connected components of B are of size at most 2, then all win lines are disjoint. Chaos can simply pass until Order moves in a win line and counter. □

For Chaos, we extend Theorem I.4.

Definition VI.2. We say $\operatorname{ovc}(B, m, p)$ has *good parity*
(i) in case B is finite, when $p = \text{Chaos}$ if and only if $|B|$ is odd.
(ii) in case B is co-finite, i.e., $\mathbb{Z}^2 \setminus B$ is finite, when $p = \text{Chaos}$ if and only if $|\mathbb{Z}^2 \setminus B|$ is odd.
(iii) in case B and $\mathbb{Z}^2 \setminus B$ are infinite, always.

Lemma VI.3. *If $B \subseteq \mathbb{Z}^2$ is neither finite nor co-finite, then there are infinitely many lines $L \subseteq \mathbb{Z}^2$ of length 2 for which $|L \cap B| = 1$.*

Proof. Suppose there are only finitely many lines $L \subseteq \mathbb{Z}^2$ of length 2 for which $|L \cap B| = 1$ and let E be their union. Then, up to a translation of B, there exists some $n \geq 0$ such that $E \subseteq [n]^2$. Since B is not finite, $B \setminus [n]^2$ is non-empty. For each $b \in B \setminus [n]^2$ all its neighbours a are in $B \setminus [n]^2$, otherwise $\{a, b\} \in E$. Since $\mathbb{Z}^2 \setminus [n]^2$ is connected we have $\mathbb{Z}^2 \setminus [n]^2 \subseteq B$, so B is co-finite. The lemma follows from contradiction. □

Theorem VI.4. *Let B be a possibly infinite board and let p be a starting player. Then*
(i) *Chaos wins $\operatorname{ovc}(B, m, p)$ for all $m \geq 9$ when the game has good parity.*
(ii) *Chaos wins $\operatorname{ovc}'(B, m, \text{Order})$ for all $m \geq 10$.*

Proof. (i) When B is finite this is Theorem I.4.i. When B is co-finite, Strategy III.2 is still applicable: there are only finitely many unmatched squares, and because of the parity, Chaos is never first to play in a pair. We never enter step (iii), so as in the proof of Theorem I.4, no homogeneous line of line of length 9 can ever exist.

Now consider the case where B is neither finite nor co-finite. Partition the lines of length 2 in $\mathbb{Z}^2$ by their image in $(\mathbb{Z}/8\mathbb{Z})^2$. At least one of these partitions $\mathcal{L}$ must contain infinitely many $L \in \mathcal{L}$ such that $|L \cap B| = 1$ by Lemma VI.3. After translating B we may assume $\mathcal{L}$ is contained in the pairing $\mathcal{P}$ given by Proposition III.1. Then, applying Strategy III.2 we have infinitely many unmatched squares, so we never enter step (iii) of the strategy, and again no homogeneous line of length 9 can occur. Thus Chaos wins $\operatorname{ovc}(B, m, p)$.

(ii) Again, the finite case was already shown. The non-finite, non-co-finite case, proceeds the same as the proof of (i), since there is an infinite number of unmatched squares. For the co-finite case, we can find a pairing of $(\mathbb{Z}/9\mathbb{Z})^2$ as in Proposition III.1 with ample unmatched squares to show that Chaos wins $\operatorname{ovc}'(\mathbb{Z}^2, 10, \text{Order})$ and thus $\operatorname{ovc}'(B, 10, \text{Order})$. □

VII. Conclusions and future research

In this paper, we proved that the game of Order versus Chaos is winning for Order if her objective is to make a line of length at most 5, and the board is suitably large. For the proof, we constructed explicit strategies, fuelled by MCTS simulations. For Chaos, we proved that he wins the game if Order needs to make a line of length at least 10, using a SAT solver to find a winning strategy. For winning lines of length between 6 and 9, we showed that Chaos wins if the board is of suitable size or parity. Furthermore, we generalized some of the theoretical results to infinite boards.

For boards which do not meet these requirements, we ran MCTS simulations to develop conjectures. The results of these simulations suggest that Order wins the game $\mathrm{ovc}([n]^2, m, \text{Order})$ if and only if n is sufficiently larger than m. Note that for general boards B and C with $B \subseteq C$, it is not necessarily the case that $\mathrm{ovc}(B, m, p)$ being won by Order implies that $\mathrm{ovc}(C, m, p)$ is won by Order. However, from the MCTS simulations, one might conjecture that this statement does hold for the games $\mathrm{ovc}([n]^2, m, \text{Order})$ with $m = 6, 7, 8$. It would be interesting to see whether this could be proven.

Besides drawing conclusions from the generated MCTS results, it might be fruitful to explore other AI techniques, such as deep learning, in order to derive more information. For the games $\mathrm{ovc}([n]^2, m, \text{Order})$ with $m = 7$ or $m = 8$, for example, based on the current results, nothing can be said on the threshold for n (if this exists) at which the game becomes winning for Order. Moreover, different techniques may show different strategic behaviour, of which the analysis may lead to new theoretical insights.

Finally, the question of whether passing is advantageous for either player is an interesting one to further investigate, it being a crucial difference between Order versus Chaos and classic maker-breaker games. In Section II, we discussed that passing once can theoretically be an advantage for both players to solve parity problems, while passing twice in a row is never necessary to win. It is unknown whether passing more often offers an advantage to either player. However, the simulations in Section V, showing that Tables I through IV are roughly the same, strongly suggest that passing is not beneficial except for in edge cases like Example II.1.

Acknowledgment

The authors would like to thank Wessel van Woerden, Walter Kosters and Flora Spieksma for their useful comments, as well as Ludo Pulles and Pim Spelier for their help in constructing the SAT instance for Proposition III.1.

Appendix

Some of our proofs rely on intensive computation. We made our source code available through GitHub at www.github.com/MadPidgeon/Order-versus-Chaos. We briefly describe the files:

`dump.cc` generates files `order.txt` and `chaos.txt` consisting of zeros and ones indicating for each of the 3^{16} game states of $\mathrm{ovc}'([4]^2, 4, \text{Chaos})$ whether it is winning for the respective player. Both of these files are of size 43 MB and can be used as input for programs to quickly verify whether a strategy is winning for a player.

`verify.py` implements an Order player using Strategy IV.2, which is verified against all possible moves of Chaos to prove Lemma IV.3.

`sat/` contains the code generating the pairing for Proposition III.1, written by Ludo Pulles and Pim Spelier.

`table/` contains the code generating the computational MCTS results for Section V.

References

[1] D. Hefetz, M. Krivelevich, M. Stojaković, and T. Szabó, *Positional games*. Springer, 2014.

[2] S. Sniderman, "Games magazine," 9 1981.

[3] B. Turner, "solved: Order wins," https://boardgamegeek.com/thread/1579397/solved-order-wins, 2016, accessed 13/3/20.

[4] D. Silver, A. Huang, C. J. Maddison, A. Guez, L. Sifre, G. van den Driessche, J. Schrittwieser, I. Antonoglou, V. Panneershelvam, M. Lanctot, S. Dieleman, D. Grewe, J. Nham, N. Kalchbrenner, I. Sutskever, T. Lillicrap, M. Leach, K. Kavukcuoglu, T. Graepel, and D. Hassabis, "Mastering the game of go with deep neural networks and tree search," *Nature*, vol. 529, pp. 484–503, 2016. [Online]. Available: http://www.nature.com/nature/journal/v529/n7587/full/nature16961.html

[5] O. Ohrimenko, P. J. Stuckey, and M. Codish, "Propagation = lazy clause generation," in *Principles and Practice of Constraint Programming – CP 2007*, C. Bessière, Ed. Springer, 2007, pp. 544–558.

[6] S. Sniderman, "order and chaos solved?" https://boardgamegeek.com/thread/1744598/order-and-chaos-solved, 2017, accessed 27/3/20.

[7] A. Biere, "PicoSAT essentials," *Journal on Satisfiability, Boolean Modeling and Computation*, vol. 4, no. 2-4, pp. 75–97, 2008.

[8] P. Erdös and J. Selfridge, "On a combinatorial game," *Journal of Combinatorial Theory, Series A*, vol. 14, no. 3, pp. 298 – 301, 1973. [Online]. Available: http://www.sciencedirect.com/science/article/pii/0097316573900058

[9] G. Chaslot, S. Bakkes, I. Szita, and P. Spronck, "Monte-Carlo tree search: A new framework for game AI." in *Artificial Intelligence for Interactive Digital Entertainment*, 2008, pp. 216–217.

[10] M. H. M. Winands, "Monte-carlo tree search," in *Encyclopedia of Computer Graphics and Games*, N. Lee, Ed. Springer, 2015, pp. 1–6. [Online]. Available: https://doi.org/10.1007/978-3-319-08234-9$_12-1$

[11] S. T. Castelein, "Order versus chaos," Bachelor's Thesis, Leiden University, 2019.